C2 Exercise 3F (changing the base of logs)

Note Title 21/02/2013

This exercise uses

$$\log_a x = \frac{\log_b x}{\log_b a}$$

and expects you to change base to $\log_{10} x$ since calculators have only recently started to have a $\log_{\blacksquare} \square$ button.

Why does this work? (Proof)

$$\log_a x = m$$

$$a^m = x$$

$$\log_b a^m = \log_b x$$

$$m \log_b a = \log_b x$$

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

x	0	1	2	3	4	5	6	7	8	9	Δ_m	1 2 3	4 5 6	7 8 9
											+		ADD	
10	·0000	0043	0086	0128	0170	0212					42	4 8 13	17 21 25	29 34 38
						0212	0253	0294	0334	0374	40	4 8 12	16 20 24	28 32 36
11	·0414	0453	0492	0531	0569	0607					39	4 8 12	16 19 23	27 31 35
						0607	0645	0682	0719	0755	37	4 7 11	15 19 22	26 30 33
12	·0792	0828	0864	0899	0934	0969					35	4 7 11	14 18 21	25 28 32
						0969	1004	1038	1072	1106	34	3 7 10	14 17 20	24 27 31
13	·1139	1173	1206	1239	1271	1303					33	3 7 10	13 16 20	23 26 30
						1303	1335	1367	1399	1430	32	3 6 10	13 16 19	22 26 29
14	·1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30	3 6 9	12 15 18	21 24 27
15	·1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28	3 6 8	11 14 17	20 22 25
16	·2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26	3 5 8	10 13 16	18 21 23
17	·2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25	2 5 7	10 12 15	17 20 22
18	·2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24	2 5 7	10 12 14	17 19 22
19	·2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22	2 4 7	9 11 13	15 18 20
20	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21	2 4 6	8 11 13	15 17 19
21	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20	2 4 6	8 10 12	14 16 18
22	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19	2 4 6	8 10 11	13 15 17
23	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18	2 4 5	7 9 11	13 14 16
24	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18	2 4 5	7 9 11	13 14 16
25	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17	2 3 5	7 9 10	12 14 15
26	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16	2 3 5	6 8 10	11 13 14
27	·4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16	2 3 5	6 8 10	11 13 14
28	·4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15	2 3 5	6 8 9	11 12 14
29	·4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15	1 3 4	6 7 9	10 12 13
30	·4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14	1 3 4	6 7 8	10 11 13
31	·4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14	1 3 4	6 7 8	10 11 13
32	·5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13	1 3 4	5 7 8	9 10 12
33	·5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13	1 3 4	5 6 8	9 10 12
34	·5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13	1 3 4	5 6 8	9 10 12
35	·5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12	1 2 4	5 6 7	8 10 11
36	·5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12	1 2 4	5 6 7	8 10 11
37	·5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12	1 2 4	5 6 7	8 10 11
38	·5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11	1 2 3	4 6 7	8 9 10
39	·5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11	1 2 3	4 6 7	8 9 10
40	·6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11	1 2 3	4 5 7	8 9 10
41	·6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10	1 2 3	4 5 6	7 8 9
42	·6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10	1 2 3	4 5 6	7 8 9
43	·6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10	1 2 3	4 5 6	7 8 9
44	·6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10	1 2 3	4 5 6	7 8 9
45	·6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10	1 2 3	4 5 6	7 8 9
46	·6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9	1 2 3	4 5 5	6 7 8
47	·6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9	1 2 3	4 5 5	6 7 8
48	·6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9	1 2 3	4 4 5	6 7 8
49	·6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9	1 2 3	4 4 5	6 7 8

	No.	log
π =	3·14159	0·49715
e =	2·71828	0·43429

$\ln x = \log_e x = (1/M) \log_{10} x$

$\log x = \log_{10} x = M \log_e x$

	No.	log
$(1/M)$ =	2·30259	0·36222
M =	0·43429	$\bar{1}$·63778

p	1	2	3	4	5	6	7	8	9	10
$\log e^p$	0·4343	0·8686	1·3029	1·7372	2·1715	2·6058	3·0401	3·4744	3·9087	4·3429
$\log e^{-p}$	$\bar{1}$·5657	$\bar{1}$·1314	$\bar{2}$·6971	$\bar{2}$·2628	$\bar{3}$·8285	$\bar{3}$·3942	$\bar{4}$·9599	$\bar{4}$·5256	$\bar{4}$·0913	$\bar{5}$·6571

x	0	1	2	3	4	5	6	7	8	9	Δ_m	1 2 3	4 5 6	7 8 9
											+		ADD	
50	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9	1 2 3	4 4 5	6 7 8
51	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8	1 2 2	3 4 5	6 6 7
52	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8	1 2 2	3 4 5	6 6 7
53	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8	1 2 2	3 4 5	6 6 7
54	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8	1 2 2	3 4 5	6 6 7
55	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8	1 2 2	3 4 5	6 6 7
56	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8	1 2 2	3 4 5	6 6 7
57	·7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8	1 2 2	3 4 5	6 6 7
58	·7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	8	1 2 2	3 4 5	6 6 7
59	·7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7	1 1 2	3 4 4	5 6 6
60	·7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7	1 1 2	3 4 4	5 6 6
61	·7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7	1 1 2	3 4 4	5 6 6
62	·7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7	1 1 2	3 3 4	5 6 6
63	·7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7	1 1 2	3 3 4	5 6 6
64	·8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7	1 1 2	3 3 4	5 6 6
65	·8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7	1 1 2	3 3 4	5 6 6
66	·8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7	1 1 2	3 3 4	5 6 6
67	·8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6	1 1 2	2 3 4	4 5 5
68	·8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6	1 1 2	2 3 4	4 5 5
69	·8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6	1 1 2	2 3 4	4 5 5
70	·8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6	1 1 2	2 3 4	4 5 5
71	·8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6	1 1 2	2 3 4	4 5 5
72	·8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6	1 1 2	2 3 4	4 5 5
73	·8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6	1 1 2	2 3 4	4 5 5
74	·8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6	1 1 2	2 3 4	4 5 5
75	·8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6	1 1 2	2 3 4	4 5 5
76	·8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6	1 1 2	2 3 4	4 5 5
77	·8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6	1 1 2	2 3 4	4 5 5
78	·8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6	1 1 2	2 3 4	4 5 5
79	·8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	6	1 1 2	2 3 4	4 5 5
80	·9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5	1 1 2	2 3 3	4 4 5
81	·9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5	1 1 2	2 3 3	4 4 5
82	·9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5	1 1 2	2 3 3	4 4 5
83	·9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5	1 1 2	2 3 3	4 4 5
84	·9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5	1 1 2	2 3 3	4 4 5
85	·9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5	1 1 2	2 3 3	4 4 5
86	·9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5	1 1 2	2 3 3	4 4 5
87	·9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5	0 1 1	2 2 3	3 4 4
88	·9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5	0 1 1	2 2 3	3 4 4
89	·9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5	0 1 1	2 2 3	3 4 4
90	·9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5	0 1 1	2 2 3	3 4 4
91	·9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5	0 1 1	2 2 3	3 4 4
92	·9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5	0 1 1	2 2 3	3 4 4
93	·9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5	0 1 1	2 2 3	3 4 4
94	·9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5	0 1 1	2 2 3	3 4 4
95	·9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5	0 1 1	2 2 3	3 4 4
96	·9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	4	0 1 1	2 2 2	3 3 4
97	·9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4	0 1 1	2 2 2	3 3 4
98	·9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4	0 1 1	2 2 2	3 3 4
99	·9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4	0 1 1	2 2 2	3 3 4

Examples 1

$$\log_2 1024 = 10$$

$$\log_2 1024 = \frac{\log_{10} 1024}{\log_{10} 2}$$

$$\log_2 1000 = 9.96578485$$

$$\log_2 1000 = \frac{\log_{10} 1000}{\log_{10} 2}$$

Examples 2

Solve $2^x = 1024$

$\log_{10} 2^x = \log_{10} 1024$

$x \log_{10} 2 = \log_{10} 1024$

$x = \frac{\log_{10} 1024}{\log_{10} 2} = 10$

Solve $2^x = 1000$

$\log_{10} 2^x = \log_{10} 1000$

$x \log_{10} 2 = 3$

$x = \frac{3}{\log_{10} 2} = 9.965\ldots$

Examples 3a - proof

We know $\log_a x = \frac{\log_b x}{\log_b a}$

Now if $x = b$ then we get...

$$\log_a b = \frac{\log_b b}{\log_b a}$$

$$\log_a b = \frac{1}{\log_b a}$$

Examples 36 switch base

$$\log_3 x + 4\log_x 3 = 5$$

$$\log_3 x + \frac{4}{\log_3 x} = 5$$

$y = \log_3 x$

$$y + \frac{4}{y} = 5$$

$$y^2 + 4 = 5$$

$$y^2 - 5x + 4 = 0$$

$$(y-4)(y-1) = 0$$

$y = 4$ or $y = 1$

$\log_3 x = 4$ or $\log_3 x = 1$

$x = 3^4$ or $x = 3^1$

$x = 81$ $\quad$ $x = 3$

Example 3c one base related to another

$$\log_3 x + \log_9 x = 5$$

$$\log_3 x + \frac{\log_3 x}{\log_3 9} = 5$$

$$\log_3 x + \frac{\log_3 x}{2} = 5$$

$$2\log_3 x + \log_3 x = 10$$

$$3\log_3 x = 10$$

$$\log_3 x = \frac{10}{3}$$

$$3^{\frac{10}{3}} = x$$

$$x = 38.94\ldots$$

1 Find to 3d.p

a $\log_7 120 = \dfrac{\log_{10} 120}{\log_{10} 7} = 2.460284071 \simeq 2.460$ to 3d.p

b $\log_3 45 = \dfrac{\log_{10} 45}{\log_{10} 3} = 3.464973521 \simeq 3.465$ to 3d.p

c $\log_2 19 = \dfrac{\log_{10} 19}{\log_{10} 2} = 4.247927513 \simeq 4.248$ to 3d.p

1d $\log_{11} 3 = \dfrac{\log_{10} 3}{\log_{10} 11} = 0.458156\,91 \simeq 0.458$ to 3 d.p.

1e $\log_6 4 =$

2a $8^x = 14$

$\Rightarrow \log_{10} 8^x = \log_{10} 14$

$\Rightarrow x \log_{10} 8 = \log_{10} 14$

$\Rightarrow x = \dfrac{\log_{10} 14}{\log_{10} 8}$

$\Rightarrow x = 1.269\,118\,307$

$x \simeq 1.27$ to 3 s.f.

2b $9^x = 99$

$\Rightarrow \log_{10} 9^x = \log_{10} 99$

$\Rightarrow x \log_{10} 9 = \log_{10} 99$

$\Rightarrow x = \dfrac{\log_{10} 99}{\log_{10} 9}$

$\Rightarrow x = 2.091329169$

$\Rightarrow x \simeq 2.09$ to 3s.f.

2c $12^x = 6$

3a $\log_2 x = 8 + 9\log_x 2$

$\Rightarrow \log_2 x = 8 + \dfrac{9}{\log_2 x}$

define $y = \log_2 x$

then

$$y = 8 + \frac{9}{y}$$

$\Rightarrow \quad y^2 = 8y + 9$

$\Rightarrow \quad y^2 - 8y - 9 = 0$

$\Rightarrow \quad (y-9)(y+1) = 0$

$\Rightarrow \quad y = 9 \quad \text{or} \quad y = -1$

$\Rightarrow \quad \log_2 x = 9 \quad \text{or} \quad \log_2 x = -1$

$\Rightarrow \quad x = 2^9 \quad \text{or} \quad x = 2^{-1}$

$\Rightarrow \quad x = 512 \quad \text{or} \quad \frac{1}{2}.$

Now since $\log_a x = \dfrac{\log_b x}{\log_b a}$

when $x = b$ we get

$$\log_a b = \frac{\log_b b}{\log_b a}$$

$$\Rightarrow \quad \log_a b = \frac{1}{\log_b a}$$

36 $\log_4 x + 2\log_x 4 + 3 = 0$

$\Rightarrow \log_4 x + \frac{2}{\log_4 x} + 3 = 0$

put $y = \log_4 x$

$\Rightarrow y^2 + 3y + 2 = 0$

$\Rightarrow (y+2)(y+1) = 0$

$\Rightarrow y = -2$ or $y = -1$

$\Rightarrow \log_4 x = -2$ or $\log_4 x = -1$

$\Rightarrow x = 4^{-2}$ or $x = 4^{-1}$

$\Rightarrow x = \frac{1}{16}$ or $x = \frac{1}{4}$

3c Solve $\log_2 x + \log_4 x = 2$